# Products, Powers, Quotients and Roots of Complex Numbers

## Product and Quotient of Complex Numbers

If $z_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right)$, and $z_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right)$, then

$$z_1 z_2 = r_1 r_2\left[\cos\left(\theta_1+\theta_2\right) + i\sin\left(\theta_1+\theta_2\right)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1-\theta_2\right) + i\sin\left(\theta_1-\theta_2\right)\right]$$

**Examples:** Find the product and quotient using trigonometric form.

$$z_1 = 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),\quad z_2 = 8\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

a) Find $z_1 z_2$

b) Find $\frac{z_1}{z_2}$

## Complex Conjugates

The conjugate of $r\left(\cos(\theta) + i\sin(\theta)\right)$ is $r\left(\cos(-\theta) + i\sin(-\theta)\right)$

A complex number times its conjugate equals $r^2$.

Proof: $r\left(\cos\theta + i\sin\theta\right)\cdot r\left(\cos(-\theta) + i\sin(-\theta)\right)$

$$= r^2\left(\cos(\theta-\theta) + i\sin(\theta-\theta)\right)$$

$$= r^2\left(\cos 0 + i\sin 0\right)$$

$$= r^2\left(1+0i\right) = r^2$$

**Example:** Find the product of the following and its conjugate: $6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

**De Moivre's Theorem**

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and $n$ is any positive integer, then

$$z^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

**Examples:**

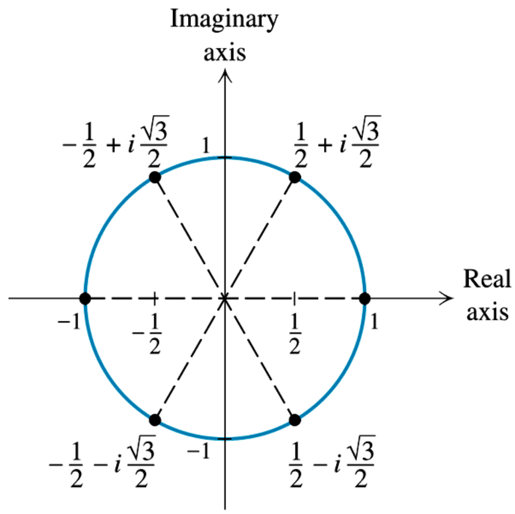
a) Simplify $(1+i)^6$.

b) Simplify $\left(\sqrt{3}-i\right)^4$.

**Roots of a Complex Number**

How many square roots does 4 have?
How many square roots does $-9$ have?
How many sixth roots does 1 have?

It turns out that 1 has 6 sixth roots, and they are distributed evenly around the complex plane.




Sixth Roots of Unity

The complex number $a+bi$ is an $n$th root of the complex number $z$ if $\left(a+bi\right)^n = z$.

For any positive integer $n$, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly $n$ distinct $n$th roots given by the expression $r^{1/n}\left(\cos\alpha + i\sin\alpha\right)$ where $\alpha = \dfrac{\theta + 360°k}{n}$ for $k = 0,1,2,\ldots,n-1$.

In radians, the roots are given by $r^{1/n}\left(\cos\alpha + i\sin\alpha\right)$ where $\alpha = \dfrac{\theta + 2k\pi}{n}$ for $k = 0,1,2,\ldots,n-1$.

The first of the $n$ roots has an argument of $\dfrac{\theta}{n}$, and the other roots are spaced $\left(\dfrac{360}{n}\right)^{\circ}$ apart.

(The circle is divided evenly into $n$ pieces.)

**Examples:**

a) Find all the fourth roots of the complex number $-8-8i\sqrt{3}$.

b) Find all the cube roots of 125.

c) Find all complex solutions to $x^3 - 27i = 0$.